

Compactly Supported Cohomology Groups of Smooth Toric Surfaces

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Abstract: This article uses homological methods for evaluating compactly supported cohomology groups of noncompact smooth toric surfaces.

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1 Introduction

Compactly supported cohomology groups play a key role in complex analysis. Vanishing properties are related to solvability of $\bar{\partial}$ problem as well as analytic continuation problems ([5] and [6]). This article uses the additive property for finding the compactly supported cohomology groups of smooth toric surfaces. The sheaf of germs of holomorphic functions on X is denoted as \mathcal{O}_X or simply \mathcal{O} , if there is no confusion which space is considered.

Compactifications and partial compactifications of toric surfaces play a key role in the procedure of computing compactly supported cohomology groups. Let X be a noncompact toric surface associated with the fan Σ . Let us denote a toric compactification of X as \tilde{X} with the fan $\tilde{\Sigma}$. The approach to the problem varies depending on convexity properties of the connected components of the (open) set $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$. The results are described in three theorems:

1. Exactly one component of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$ is concave (Theorem 3.1). This condition is equivalent to the fan Σ being a subfan of a strictly convex fan.
2. Exactly one component of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$ is a half-plane (Theorem 3.2). This condition is equivalent to Σ being a subfan of a fan that covers a half plane.
3. Exactly two components of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$ are half-planes (Example 3.1). This condition implies that the toric surface is $\mathbb{P}^1 \times \mathbb{C}^1$.
4. All components of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$ are strictly convex (Theorem 3.3). It can be seen as the fan Σ spans the entire plane.

In this terminology a component is strictly convex if it does not contain a line.

An introduction to the theory of toric varieties can be found in [1] or in [4]. Toric surfaces are presented shortly in [12].

2 Compactly Supported Cohomology Groups

This section contains a short overview of definitions and main properties of the compactly supported cohomology groups.

2.1 Definition and Examples

Definition 2.1 (Compactly supported Dolbeault cohomology groups) *Compactly supported Dolbeault cohomology groups of the domain D are the complex vector spaces:*

$$\mathfrak{H}_c^{p,q}(D) = \frac{\{\bar{\partial}\text{-closed forms with compact support of bidegree } (p, q) \text{ in } D\}}{\{\bar{\partial}\text{-exact forms with compact support of bidegree } (p, q) \text{ in } D\}}.$$

The following theorem shows the relationship between compactly supported cohomology groups and compactly supported Dolbeault cohomology groups.

Theorem 2.1 (Dolbeault's Theorem, [3]) *If D is an open domain in the space of n complex variables, \mathcal{O} is the sheaf of germs of holomorphic functions on D , and $\mathfrak{H}_c^{p,q}(D)$ is compactly supported Dolbeault cohomology group of bidegree (p, q) for D . Then $H_c^q(D, \mathcal{O}) = \mathfrak{H}_c^{0,q}(D)$. ■*

An alternate definition can be found in [2]. In particular, if X is a compact manifold, then

$$H_c^i(X, \mathcal{O}) = H^i(X, \mathcal{O}) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{C} & i = 0, \end{cases}$$

i.e., compactly supported groups are equal to the usual cohomology groups for compact spaces. It is a well known fact that the groups $H_c^i(\mathbb{C}^n, \mathcal{O})$ are trivial for all $n \geq 2$ and $i \geq 0$ ([11]).

2.2 Properties

Compactly supported cohomology groups have all properties of a cohomology theory. The “additive” property, broadly used in the further part of the research, requires the notion of the inverse image of a sheaf.

Definition 2.2 (Inverse image) *Let $f : A \rightarrow B$ be a map and let \mathcal{G} be a sheaf on B with canonical projection $\pi : \mathcal{G} \rightarrow B$. The inverse image sheaf $f^*\mathcal{G}$ is defined as*

$$f^*\mathcal{G} = \{(a, g) \in A \times \mathcal{G} : f(a) = \pi(g)\}.$$

In particular, if f is a closed embedding, the following theorem holds.

Theorem 2.2 ([10] III.7.6) *Let $i : Y \rightarrow X$ be a closed embedding, then the following sequence*

$$\dots \rightarrow H_c^q(X \setminus Y, \mathcal{F}) \rightarrow H_c^q(X, \mathcal{F}) \rightarrow H_c^q(Y, i^*\mathcal{F}) \rightarrow H_c^{q+1}(X \setminus Y, \mathcal{F}) \rightarrow \dots,$$

is exact. ■

The following exact sequences are obtained for each n separately.

Theorem 2.3 (The Künneth Formula, [2] II. Theorem 15.2) *If X and Y are locally compact Hausdorff spaces, with the sheaves \mathcal{F} and \mathcal{G} respectively and $\mathcal{F} * \mathcal{G} = 0$, then the sequence*

$$0 \rightarrow \bigoplus_{p+q=n} H_c^p(X, \mathcal{F}) \otimes H_c^q(Y, \mathcal{G}) \rightarrow H_c^n(X \times Y, \mathcal{F} \otimes \mathcal{G}) \rightarrow \bigoplus_{p+q=n+1} H_c^p(X, \mathcal{F}) * H_c^q(Y, \mathcal{G}) \rightarrow 0$$

is exact.

3 Compactly Supported Cohomology Groups of Toric Surfaces

Main results are written in the form of three theorems, depending on if the support of the fan of the toric surface spans less than a half plane, a half-plane, or the entire plane. The case when the fan spans a line requires entirely different approach and is described in Example 3.1. All previous cases use a compactification or a partial compactification of the toric surface. If the fan Σ is a subfan of a fan that is strictly convex or covers a half plane then a partial compactification will be sufficient. For all cases the notation is the same.

Let X be a smooth noncompact toric variety associated with the fan Σ and let its toric compactification \tilde{X} be associated with the fan $\tilde{\Sigma}$. Clearly, Σ is a subfan of $\tilde{\Sigma}$ and we can consider components of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$. Let C_1, \dots, C_n be strictly convex components of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$ and let C_0 be the concave component or an (open) half plane (the component C_0 can be omitted since it does not appear in the cohomology formula). Then $\tilde{X} \setminus X$ consists of connected components Y_1, \dots, Y_n that are defined by the components C_1, \dots, C_n . The embeddings of Y_j into \tilde{X} will be denoted as $i_j : Y_j \rightarrow \tilde{X}$ and the (strictly) convex connected components of $\text{supp}(\tilde{\Sigma}) \setminus \text{supp}(\Sigma)$ are spanned by the pairs of vectors v_j and w_j (with the positive orientation of \mathbb{R}^2) and have the (singularity) type (p_j, q_j) as described for toric surfaces in [9] section 2.6.

3.1 The Fan Spans Less than a Half-Plane

The following theorem describes compactly supported cohomology groups of a smooth toric surface X with a fan Σ that spans less than a half plane. In other words, Σ can be seen as a subfan of a strictly convex fan.

Theorem 3.1 *Let X be a smooth toric surface which fan Σ is a subfan of a strictly convex fan. Then $H_c^0(X, \mathcal{O}) = H_c^2(X, \mathcal{O}) = 0$ and*

$$H_c^1(X, \mathcal{O}) = \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O}) = \bigoplus_{j=1}^n \left\{ \sum_{(s,t) > (0,0)} a_{st} z_j^s w_j^t : p_j t \geq q_j s \right\},$$

where Y_j are varieties defined by (strictly) convex connected components of $\mathbb{R}^2 \setminus \text{supp}(\Sigma)$ and i_j are (closed) embeddings $i_j : Y_j \rightarrow X$ and the series converge.

Proof. Since Σ is a subfan of a strictly convex fan, X can be treated as an open submanifold of a smooth toric surface \tilde{X} with a strictly convex fan $\tilde{\Sigma}$. Then $X = \tilde{X} \setminus Y$, where Y is a subvariety of \tilde{X} , and $Y = Y_1 \cup \dots \cup Y_n$ is its decomposition into disjoint, compact and connected subvarieties. Note that the additive property provides the following exact sequence.

$$0 \rightarrow H_c^0(X, \mathcal{O}) \rightarrow H_c^0(\tilde{X}, \mathcal{O}) \rightarrow H_c^0(Y, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow H_c^1(\tilde{X}, \mathcal{O}) \rightarrow H_c^1(Y, i^* \mathcal{O}) \rightarrow H_c^2(X, \mathcal{O}) \rightarrow H_c^2(\tilde{X}, \mathcal{O}) \rightarrow 0.$$

Then $H_c^0(X, \mathcal{O}) = H_c^0(\tilde{X}, \mathcal{O}) = 0$ because X and \tilde{X} are noncompact; $H_c^0(Y, i^* \mathcal{O}) = H_c^0(Y, \mathcal{O}_Y) = \bigoplus_{j=1}^n H_c^0(Y_j, \mathcal{O}_{Y_j})$ because Y_j are compact and disjoint; $H_c^1(\tilde{X}, \mathcal{O}) = 0$ from Theorem 4.2 in [12]; and $H_c^2(\tilde{X}, \mathcal{O}) = 0$ because of dimensional reasons. Thus the sequence simplifies to the following two sequences:

$$0 \rightarrow H_c^0(Y, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow 0$$

and

$$0 \rightarrow H_c^1(Y, i^* \mathcal{O}) \rightarrow H_c^2(X, \mathcal{O}) \rightarrow H_c^2(\tilde{X}, \mathcal{O}) \rightarrow 0.$$

Then

$$H_c^1(X, \mathcal{O}) = H_c^0(Y, i^* \mathcal{O}) = \bigoplus_{j=1}^n H_c^0(Y_j, i^* \mathcal{O}),$$

since Y_j are disjoint and compact components of Y and

$$H_c^2(X, \mathcal{O}) = H_c^1(Y, i^* \mathcal{O}) = \bigoplus_{j=1}^n H_c^1(Y_j, i^* \mathcal{O}) = 0,$$

since each Y_j is compact (as the sum of projective curves) in \tilde{X} . ■

Note that the groups $H_c^0(Y_j, i^* \mathcal{O})$ can be found explicitly in terms of p_j and q_j . If (z_j, w_j) are local coordinates then

$$H_c^0(Y_j, i^* \mathcal{O}) = \left\{ \sum_{(s,t) > (0,0)} a_{st} z_j^s w_j^t : p_j t \geq q_j s \right\},$$

where the series $\sum_{(s,t) > (0,0)} a_{st} z_j^s w_j^t$ converges in the coordinates (z_j, w_j) in some neighborhood of Y_j in \tilde{X} . The details can be found in Section 2.2 of [13].

3.2 The Fan Spans a Half-Plane

The following lemma evaluates the compactly supported cohomology groups for those toric surfaces which fans have the supports that covers a half-plane. We will need this results for the next theorem.

Lemma 3.1 *Let X be a smooth toric surface which fan Σ such that $\text{supp}(\Sigma)$ is a half-plane. Then $H_c^0(X, \mathcal{O}) = H_c^2(X, \mathcal{O}) = 0$ and $H_c^1(X, \mathcal{O}) = \{\sum_{s>0} a_s z^s\}$, where the series converges in a neighborhood of 0.*

Proof. Note that that X can be represented as $X = \tilde{X} \setminus \mathbb{P}^1$, where \tilde{X} is a smooth compact toric surface. Then X admits the following exact sequence:

$$0 \rightarrow H_c^0(X, \mathcal{O}) \rightarrow H_c^0(\tilde{X}, \mathcal{O}) \rightarrow H_c^0(\mathbb{P}^1, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow H_c^1(\tilde{X}, \mathcal{O}) \rightarrow H_c^1(\mathbb{P}^1, i^* \mathcal{O}) \rightarrow H_c^2(X, \mathcal{O}) \rightarrow H_c^2(\tilde{X}, \mathcal{O}) \rightarrow 0.$$

Note that $H_c^0(X, \mathcal{O}) = 0$ since X is noncompact and $H_c^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ since \tilde{X} is compact. Moreover $H_c^1(\tilde{X}, \mathcal{O}) = H_c^1(\mathbb{P}^1, i^* \mathcal{O}) = H_c^2(X, \mathcal{O}) = H_c^2(\tilde{X}, \mathcal{O}) = 0$ and the sequence simplifies to:

$$0 \rightarrow \mathbb{C} \rightarrow H_c^0(\mathbb{P}^1, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow 0.$$

Then the group $H_c^1(X, \mathcal{O})$ is a quotient of $H_c^0(\mathbb{P}^1, i^* \mathcal{O})$ and \mathbb{C} . Since the embedding of \mathbb{P}^1 into \tilde{X} is flat we obtain that $H_c^0(\mathbb{P}^1, i^* \mathcal{O}) = \{\sum_{s \geq 0} a_s z^s\}$ and $H_c^1(X, \mathcal{O}) = \{\sum_{s>0} a_s z^s\}$, where the series converge in a neighborhood of 0. ■

Note that $H_c^1(X, \mathcal{O}) = H_c^1(\mathbb{C}^1, \mathcal{O})$.

Now the following theorem can be formulated.

Theorem 3.2 *Let X be a smooth toric surface which fan Σ is a subfan of a fan $\tilde{\Sigma}$ which support is a half-plane. Then $H_c^0(X, \mathcal{O}) = H_c^2(X, \mathcal{O}) = 0$ and*

$$H_c^1(X, \mathcal{O}) = H_c^1(\mathbb{C}^1, \mathcal{O}) \oplus \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O}) = \left\{ \sum_{s>0} a_s z^s \right\} \oplus \bigoplus_{j=1}^n \left\{ \sum_{(s,t) > (0,0)} a_{st} z^s w^t : p_j t \geq q_j s \right\},$$

where all series converge and the (strictly) convex connected components of $\text{supp} \tilde{\Sigma} \setminus \text{supp}(\Sigma)$ define the varieties Y_i .

Proof. Let $\tilde{X} \setminus X = Y$, where \tilde{X} is a smooth compact toric variety and $Y = Y_1 \cup \dots \cup Y_n$ is the decomposition of Y into compact and connected components. Then

$$0 \rightarrow H_c^0(X, \mathcal{O}) \rightarrow H_c^0(\tilde{X}, \mathcal{O}) \rightarrow H_c^0(Y, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow H_c^1(\tilde{X}, \mathcal{O}) \rightarrow H_c^1(Y, i^* \mathcal{O}) \rightarrow H_c^2(X, \mathcal{O}) \rightarrow H_c^2(\tilde{X}, \mathcal{O}) \rightarrow 0.$$

Then: $H_c^0(X, \mathcal{O}) = H_c^0(\tilde{X}, \mathcal{O}) = 0$ because X and \tilde{X} are noncompact; $H_c^0(Y, i^* \mathcal{O}) = \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O})$ because Y_j are compact and disjoint; $H_c^1(\tilde{X}, \mathcal{O}) = \{\sum_{s>0} a_s z^s\}$ from Lemma 3.1; $H_c^2(\tilde{X}, \mathcal{O}) = 0$ from Lemma 3.1; and $H_c^1(Y, i^* \mathcal{O}) = 0$ as in the proof of the previous theorem. Thus the sequence simplifies to:

$$0 \rightarrow H_c^0(Y, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow H_c^1(\tilde{X}, \mathcal{O}) \rightarrow 0 \rightarrow H_c^2(X, \mathcal{O}) \rightarrow 0.$$

Then $H_c^2(X, \mathcal{O}) = 0$ and

$$0 \rightarrow \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow H_c^1(\tilde{X}, \mathcal{O}) \rightarrow 0.$$

The sequence does not split but we still obtain the following result

$$H_c^1(X, \mathcal{O}) = H_c^1(\mathbb{C}^1, \mathcal{O}) \oplus \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O}) = \left\{ \sum_{s>0} a_s z_0^s \right\} \oplus \bigoplus_{j=1}^n \left\{ \sum_{(s,t)>(0,0)} a_{st} z_j^s w_j^t : p_j t \geq q_j s \right\}.$$

■

The details of the last step are presented in Section 2.2 of [13].

3.3 The Fan Spans a Line

If the support of the fan Σ is a line, then the toric surface is simply $\mathbb{P}^1 \times \mathbb{C}^*$ and its cohomology groups can be computed from the Künneth formula.

Example 3.1 *Let us find $H_c^i(\mathbb{P}^1 \times \mathbb{C}^*, \mathcal{O})$ for $i = 0, 1, 2$ using the Künneth formula for products. Recall that $H_c^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$, $H_c^1(\mathbb{P}^1, \mathcal{O}) = 0$, $H_c^0(\mathbb{C}^*, \mathcal{O}) = 0$ and $H_c^1(\mathbb{C}^*, \mathcal{O}) = \left\{ \sum_{s>0} a_s z^s, a_s \in \mathbb{C} \right\} \oplus \left\{ \sum_{s>0} a_s \frac{1}{z^s}, a_s \in \mathbb{C} \right\}$. Then the Künneth Formula for $\mathbb{P}^1 \times \mathbb{C}^*$ gives the following for the first cohomology group of $\mathbb{P}^1 \times \mathbb{C}^*$:*

$$0 \rightarrow \bigoplus_{p+q=1} H_c^p(\mathbb{C}^1, \mathcal{O}) \otimes H_c^q(\mathbb{P}^1, \mathcal{O}) \rightarrow H_c^1(\mathbb{C}^* \times \mathbb{P}^1, \mathcal{O}) \rightarrow \bigoplus_{p+q=2} H_c^p(\mathbb{C}^*, \mathcal{O}) * H_c^q(\mathbb{P}^1, \mathcal{O}) \rightarrow 0,$$

which converts to:

$$0 \rightarrow H_c^1(\mathbb{C}^*, \mathcal{O}) \bigoplus H_c^0(\mathbb{P}^1, \mathcal{O}) \rightarrow H_c^1(\mathbb{C}^* \times \mathbb{P}^1, \mathcal{O}) \rightarrow H_c^1(\mathbb{C}^*, \mathcal{O}) * H_c^1(\mathbb{P}^1, \mathcal{O}) \rightarrow 0,$$

and proves that $H_c^1(\mathbb{C}^* \times \mathbb{P}^1, \mathcal{O}) = H_c^1(\mathbb{C}^*, \mathcal{O}) = \left\{ \sum_{s>0} a_s z^s \right\} \oplus \left\{ \sum_{s>0} a_s \frac{1}{z^s}, a_s \in \mathbb{C} \right\}$. Similarly for $H_c^2(\mathbb{C}^* \times \mathbb{P}^1, \mathcal{O})$:

$$0 \rightarrow H_c^1(\mathbb{C}^*, \mathcal{O}) \otimes H_c^1(\mathbb{P}^1, \mathcal{O}) \rightarrow H_c^2(\mathbb{C}^* \times \mathbb{P}^1, \mathcal{O}) \rightarrow 0,$$

which implies $H_c^2(\mathbb{C}^* \times \mathbb{P}^1, \mathcal{O}) = 0$.

3.4 The Fan Spans the Entire Plane

If the support of the fan Σ is not a subset of a half-plane then it spans the entire plane. In this case all components of $\text{supp} \tilde{\Sigma} \setminus \text{supp} \Sigma$ are strictly convex.

Theorem 3.3 *Let X be a noncompact toric surface whose fan $\Sigma \subset \mathbb{R}^n$ has the support that is not a subset of a half-plane. Then $H_c^0(X, \mathcal{O}) = H_c^2(X, \mathcal{O}) = 0$ and*

$$H_c^1(X, \mathcal{O}) = \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O}) \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O}) = \bigoplus_{j=1}^n \left\{ \sum_{(s,t)>(0,0)} a_{st} z_j^s w_j^t : p_j t \geq q_j s \right\},$$

where strictly convex connected components of $\text{supp} \tilde{\Sigma} \setminus \text{supp}(\Sigma)$ define the varieties Y_j .

Proof. Let $\tilde{X} \setminus X = Y$, where \tilde{X} is a smooth compact toric variety and $Y = Y_1 \cup \dots \cup Y_n$ is the decomposition of Y into compact and connected components. Then the additive property has the following form:

$$0 \rightarrow H_c^0(X, \mathcal{O}) \rightarrow H_c^0(\tilde{X}, \mathcal{O}) \rightarrow H_c^0(Y, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow H_c^1(\tilde{X}, \mathcal{O}) \rightarrow H_c^1(Y, i^* \mathcal{O}) \rightarrow H_c^2(X, \mathcal{O}) \rightarrow H_c^2(\tilde{X}, \mathcal{O}) \rightarrow 0.$$

Recall that $H_c^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ and $H_c^1(\tilde{X}, \mathcal{O}) = H_c^2(\tilde{X}, \mathcal{O}) = 0$ since \tilde{X} is compact. Moreover, since X is noncompact, $H_c^0(X, \mathcal{O}) = 0$. Thus the sequence can be written as two sequences:

$$0 \rightarrow \mathbb{C} \rightarrow H_c^0(Y, i^* \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O}) \rightarrow 0$$

and

$$0 \rightarrow H_c^1(Y, i^* \mathcal{O}) \rightarrow H_c^2(X, \mathcal{O}) \rightarrow 0.$$

Then $H_c^2(X, \mathcal{O}) = H_c^1(Y, i^* \mathcal{O})$ and

$$H_c^1(X, \mathcal{O}) = H_c^0(Y, i^* \mathcal{O})/\mathbb{C} = \bigoplus_{j=1}^n H_c^0(Y_j, i_j^* \mathcal{O})/\mathbb{C} = \bigoplus_{j=1}^n \left\{ \sum_{(s,t) > (0,0)} a_{st} z_j^s w_j^t : p_j t \geq q_j s \right\}$$

■

4 Further Research

If $\Sigma \subset \mathbb{R}^m$ is a fan associated with a noncompact toric variety X then the compactly supported cohomology groups as well as the obstacles for solving extension problems lie in the connected components of $\mathbb{R}^m \setminus \text{supp}(\Sigma)$. In terms of the geometry (or topology) of X those components describe the ends of X . The notion of an end was formally introduced by Freudenthal in [8] and is commonly used by analysts, for example in [7] or [12]. Asking which properties of ends in higher dimensions have impact on the cohomology groups yields to interesting questions that involve compactifications of toric varieties and tropical geometry.

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